Short note concerning neutrino mass eigenstates

(Private Research-Associates*), int. Paper, dated: July 2026)

In our search of hypothetical, fractionally charged heavy leptons (see the summary article PDF "Fractionally charged massive (FCM) leptons predictions: Summary " (2018) of the www.culetto.at website) using the M_P/m_e, M_P/m_i ($i = e, \mu, \tau$, and M_P & m_e being the Planck and electron mass, respectively) approximation formulae Eqs.(1) - (3), an anomaly as to the energy scale had been met. By extending Eq.(3) to continuous external angle values ξ_0 ex [P, 1/2] in the Mandelbrot set's chaotic c-region, and further "analytic continuation" to ξ_0 values > 1/2 and to winding number 1 too, the $2(M_P/2m_e)(m_i/M_P)(m_{e,\, exp})$ [TeV/c^2] relation yielded m_i-results around 5TeV/c^2 at $\xi_{0i} = (1 + 16/31)$. In stark contrast, winding number 1 plus the upper external angle 15/31 of the M-antenna's period-5 last appearance cardioid's cusp gave the extremely small mass result of approx.1.29 x10^{-20} TeV/c^2. Thus, thinking of some connection to light dark matter or/and neutrinos wasn't that weird.

$$\frac{M_P}{2m_e} \approx \frac{\sqrt{2}\,\ln(\delta_{2D})}{\sqrt{\pi P}|c_D|\ln(\delta)} \exp(\gamma^{1/2} e^{\pi/2+1/2} \pi^{e/2+1/2}) \qquad \text{Eq.(1)}$$

$$\frac{M_P}{m_i} \approx \frac{2\ln(\delta_{2D})}{\sqrt{\xi_{0i}P}\,\Gamma(\xi_{0i})|c_D|\ln(\delta)} \exp(\gamma^{1/2} e^{B(1/2,\,\xi i)/2+1/2} B(1/2,\, \xi_i)^{e/2+1/2}) \,,\ \ i = e, \mu, \tau \qquad \text{Eq.(2)}$$

$$\xi_i = \left(1/2 + \frac{\pi^2\Gamma(\pi/2 + 1/2)^2\xi_{0i}{}^2}{4\Gamma((e/2 + 1/2)\ln(\pi))\Gamma(1/2 - \xi_{0i})}\right) \,, \qquad \text{Eq.(3)}$$

$\xi_{0e} = 1/2$, $\xi_{0\mu}$ = SQRT(P/2), $\xi_{0\tau} = P$,

where γ is the Euler-Mascheroni constant, P the Thue-Morse constant, c_D the M set's main bifurcation-series' Myrberg-Feigenbaum point's coordinate, δ Feigenbaum's universal number, δ_{2D} the Feigenbaum number for an area-preserving 2-dimensional map (M. Tabor; E.W. Weisstein /Wolfram Research), and B being Euler's Beta function.

Like in case of the *neutron-electron mass ratio* approximation (see the corresponding PDF of our website), with the free, unstable udd quark composite's ties to chaos – dealt with via the inclusion of $M_{3,1}$ (see Pastor et al.), being the pre-period-3 and period-1 Misiurewicz point at c = –1.5436890... (= c_{GT} in Eq.(4), abbreviation for this Großmann-Thomae's B_0 - B_1 chaotic band merging point) – the same recipe was taken for the flavour-sharing neutrino partners of the mu and the tau. As to the lowest mass eigenvalue of the propagating neutrinos, the corresponding external angle ξ_{m1} is assumed the minimum of $\xi(\xi_{0i})$ for $\xi_{01} = 1/2$, the "zero point" external angle value being δ/π, the ratio of the ultimate characteristic numbers in

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period-doubling-cascade- and period-1-oscillations. Because of the quite strong dependence of the mass results on $\xi(\xi_{0i})$ – and an upper mass limit of one eV/c^2 or even less not to be trespassed – the second term of Eq.(4) had to be much smaller than the one in the $m_{e,\,\mu\,,\tau}$ context (Eq.(3)). And a residue of self-similarity, this visible in the formulae above, should make it to the new context, like angle ξ_{0i} , (1/2 – ξ_{0i}) or maybe denominator's pre-factor 4. Because lower external angles are dealt with in the FCM leptons prediction formalism, a term containing (1 – P), the lower external angle of the main bifurcation series' Myrberg-Feigenbaum point at c_D, could appear too. As pre-period-3 oscillations likely are part of the game (and the external angle values of the main series' first bifurcation root have 3 = 2^1+1 as their denominator), a 3 might also show up. By following these indications, and after a lot of trial-and-error steps in search of a suitable functional shape, an approximate formulation of the neutrino mass eigenstates' likely external angles accessory was found:

$$\xi_i = \delta/\pi + \left[\frac{(\gamma - 1)c_{GT}}{4(\Gamma(1 - P)/3 - \xi_{0i})\Gamma(1/2 - \xi_{0i})}\right]^3 , \qquad \text{Eq.(4)}$$

$\xi_{01} = 1/2$, $\xi_{02} = \mathrm{SQRT}(P/2)$, $\xi_{03} = P$

i	ξ_i	m_i [TeV/c^2]	m_i [eV/c^2]	$m^2_{i+1} - m^2_i$ [eV/c^2]^2
1	1.486253_0	$6.099_3 \times 10^{-18}$	$6.099_3 \times 10^{-6}$	0.0000751_2 $(7.41^{+0.021}_{-0.020} \times 10^{-5})$*)
2	1.489459_7	$8.667_5 \times 10^{-15}$	$8.667_5 \times 10^{-3}$	0.002333_6 $(2.437^{+0.028}_{-0.027} \times 10^{-3})$*)
3	1.490301_0	$4.907_0 \times 10^{-14}$	$4.907_0 \times 10^{-2}$	$\{\Delta m^2_{31}\} = 0.002407_9$

*) Navas, S. et al. (Particle Data Group): Review of Particle Physics, Phys. Rev. D **110**, 030001 (2024)

Table: Approximated neutrino mass eigenvalues, states' external angles accessory ξ_i, Δm^2_{21}, Δm^2_{32} , these compared to Particle Data Group's values, and Δm^2_{31}

Our calculated Δm^2 results are ± off the values cited, maybe in parts due to our procedure operating on the TeV-scale and the mass eigenvalues searched for being tiny. Furthermore, an imperfect $\xi(\xi_0)$ dependence at pretty steep $\partial m/\partial\xi(\xi_0)$ could too have contributed. A remarkable result – if not just accidental – would be a simpler (effective) neutrino flavour eigenstate - mass eigenstate link than the mixing matrix is, this time via the charged leptons' ξ_{0i}, whose number is limited to 3: the external angle 1/2 of Mandelbrot set "antenna"'s end at c = –2, the upper external angle P of the main bifurcation series' Myrberg-Feigenbaum point at c_D = – 1.401155…+ 0i, and the geometric mean of these.